Bifurcation Behaviour of the Travelling Wave Solutions of the Perturbed Nonlinear Schrödinger Equation with Kerr Law Nonlinearity

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In this paper, we study the bifurcations and dynamic behaviour of the travelling wave solutions of the perturbed nonlinear Schrödinger equation (NLSE) with Kerr law nonlinearity by using the theory of bifurcations of dynamic systems. Under the given parametric conditions, all possible representations of explicit exact solitary wave solutions and periodic wave solutions are obtained.

Key words: NLSE; Kerr Law Nonlinearity; Bifurcation; Travelling Wave Solutions.
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1. Introduction

In the recent years, many direct methods have been developed to construct travelling wave solutions to the nonlinear partial differential equations (NLPDEs), such as the trigonometric function series method [1, 2], the modified mapping method and the extended mapping method [3], the modified trigonometric function series method [4], the dynamical system approach and the bifurcation method [5], the exp-function method [6], the multiple exp-function method [7], the transformed rational function method [8], the symmetry algebra method (consisting of Lie point symmetries) [9], the Wronskian technique [10], and so on. In addition they are efficient alternative methods for solving fractional differential equations, see [11 – 13].

In this paper, we investigate the perturbed NLSE with Kerr law nonlinearity [2]

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + \alpha|u|^2u + \mathrm{i}[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x \\ &+ \gamma_3(|u|^2)_x u] = 0, \end{aligned} \tag{1}$$

where γ_1 is the third-order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is also a version of nonlinear dispersion. More details are presented in [1]. It must be very clear that γ_3 is not Raman scattering in general, but only if γ_3 is purely imaginary. Moreover, Raman scattering is not a Hamiltonian perturbation and therefore it is not an integrable perturbation. More details are presented in [4]. Equation (1) describes the propagation of optical solitons in nonlinear optical fibers that exhibits a Kerr law nonlinearity. Recently, there are lots of contributions about (1) (see for instance [2, 4, 5, 14 – 20] and references therein). Equation (1) has important applications in various fields, such as semiconductor materials, optical fiber communications, plasma physics, fluid and solid mechanics. More details are presented in [21].

It is worth mentioning that Zhang et al. [3 – 5, 14] considered the NLSE (1) with Kerr law nonlinearity and obtained some new exact travelling wave solutions of (1). In [3], by using the modified mapping method and the extended mapping method, Zhang et al. derived some new exact solutions of (1), which are the linear combination of two different Jacobi elliptic functions and investigated the solutions in the limit cases. In [4], by using the modified trigonometric function series method, Zhang et al. studied some new exact travelling wave solutions of (1). In [5], by using the dynamical system approach, Zhang et al. obtained the travelling wave solutions in terms of bright and dark optical solitons and cnoidal waves. The authors found that (1) has only three types of bounded travelling wave solutions, namely, bell-shaped solitary wave solutions, kink-shaped solitary wave solutions, and Jacobi elliptic function periodic solutions. Moreover, we pointed out the region in which these periodic wave solutions lie. We show the relation between the bounded travelling wave solution and the energy level h. We observe that these periodic wave solutions tend to the corresponding solitary wave solutions as h increases or decreases. Finally, for some special selections of the energy level h, it is shown that the exact periodic solutions evolute into solitary wave solutions. In [14], by using the modified $(\frac{G'}{G})$-expansion method, Miao and Zhang obtained the travelling wave solutions, which are expressed by hyperbolic functions, trigonometric functions, and rational functions.

In the absence of γ_1, γ_2, γ_3 (i.e. $\gamma_1 = \gamma_2 = \gamma_3 = 0$), (1) reduces to

$$iu_t + u_{xx} + \alpha|u|^2 u = 0. \tag{2}$$

It is well known that the NLSE (2) admits the bright soliton solution [22] or [4, pp. 2]

$$u(x,t) = k\sqrt{\frac{2}{\alpha}}\operatorname{sech}(k(x-2\mu t))\,\mathrm{e}^{\mathrm{i}[\mu x-(\mu^2-k^2)t]}$$

for the self-focusing case $\alpha > 0$, where α and k are arbitrary real constants, and the dark soliton solution [23] or [4, pp. 2]

$$u(x,t) = k\sqrt{-\frac{2}{\alpha}}\tanh(k(x-2\mu t))\,\mathrm{e}^{\mathrm{i}[\mu x-(\mu^2+2k^2)t]}$$

for the de-focusing case $\alpha < 0$, where α and k are arbitrary real constants. For related problems, we refer to [24, 25] and the references therein.

In [26], Kodama considered the perturbed higher-order nonlinear Schrödinger equation

$$\begin{aligned}\frac{\partial\Psi}{\partial z} = \mathrm{i}\alpha_1\frac{\partial^2\Psi}{\partial t^2} + \mathrm{i}\alpha_2|\Psi|^2\Psi + \alpha_3\frac{\partial^3\Psi}{\partial t^3}\\ + \alpha_4\frac{\partial\Psi|\Psi|^2}{\partial t} + \alpha_5\Psi\frac{\partial|\Psi|^2}{\partial t},\end{aligned} \tag{3}$$

where Ψ is a slowly varying envelop of the electric field, the subscripts z and t are the spatial and temporal partial derivative in retard time coordinates, and α_1, α_2, α_3, α_4, and α_5 are real parameters related to group velocity, self-phase modulation, third-order dispersion, self-steepening, and self-frequency shift arising from stimulated Raman scattering, respectively. Since α_5 is real valued, this perturbation term represents a nonlinear dispersion.

Recently, Liu [27] obtained some new exact travelling wave solutions of (3) by using the generally projective Riccati equation method. In [28], by using the extended Jacobi elliptic function expansion methods, El-Wakil and Abdou investigated new exact travelling wave solutions of (3) which include a new solitary or shock wave solution and envelope solitary and shock wave solutions. Later on, by using the generalized auxiliary equation method, Abdou [29] studied (3) and obtained some new types of exact travelling wave solutions, including soliton-like solutions, trigonometric function solutions, exponential solutions, and rational solutions.

However, in our contribution, we investigate the bifurcations and dynamic behaviour of travelling wave solutions of the NLSE (1) with Kerr law nonlinearity by using the theory of bifurcations of dynamic systems. Furthermore, under the given parametric conditions, we obtain all possible representations of explicit exact solitary wave solutions and periodic wave solutions.

Remark 1.1. There are discussions on more exact solutions of the standard nonlinear Schrodinger equation in [9], which the authors should better mention while discussing the bright and dark solitons of the equation. On exact solutions to soliton equations, there is a new and interesting discovery recently presented on the basis of the linear superposition principle. More details are present in [32]. This even gives linear subspaces of solutions to nonlinear equations.

2. Bifurcations and Exact Travelling Wave Solutions

In this section, we will investigate the profiles of the travelling wave solutions and give all possible exact explicit parametric representations for the bounded travelling wave solutions.

Assume that (1) has travelling wave solutions in the form [2]

$$u(x,t) = \phi(\xi)\exp(\mathrm{i}(Kx-\Omega t)),\quad \xi = k(x-ct), \tag{4}$$

where c is the propagating wave velocity.

Substituting (4) into (1) yields

$$\begin{aligned}&\mathrm{i}(\gamma_1 k^3\phi''' - 3\gamma_1 K^2 k\phi' + \gamma_2 k\phi^2\phi' + 2\gamma_3 k\phi^2\phi' - ck\phi'\\ &+ 2Kk\phi') + (\Omega\phi + k^2\phi'' - K^2\phi + \alpha\phi^3 + 3\gamma_1 Kk^2\phi''\\ &+ \gamma_1 K^3\phi - \gamma_2 K\phi^3) = 0,\end{aligned}$$

where γ_i $(i = 1,2,3)$, α, k are positive constants and the prime meaning differentiation with respect to ξ.

By virtue of [2, pp. 3065], we have

$$A\phi''(\xi)+B\phi(\xi)+C\phi^3(\xi)=0(A\neq 0), \tag{5}$$

where $A=\gamma_1 k^2$, $B=2K-c-3\gamma_1 K^2$, and $C=-\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3$.

Indeed, (5) is the well known Duffing equation which is the equation governing the oscillations of a mass attached to the end of a spring transmitting tension (or compression) [30].

Let $x=\phi(\xi)$ and $y=\phi'(\xi)$, then (5) reduces to the following planar dynamic system:

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\xi}&=y,\\ \frac{\mathrm{d}y}{\mathrm{d}\xi}&=-\frac{B}{A}\phi+\frac{C}{A}\phi^3.\end{aligned} \tag{6}$$

For simplicity, we assume $\beta=\frac{B}{A}$, $\gamma=-\frac{C}{A}$. Then (6) has the Hamiltonian function

$$H(\phi,y)=\frac{1}{2}y^2+\frac{1}{2}\beta\phi^2+\frac{1}{4}\gamma\phi^4=h, \tag{7}$$

where $h\in\mathbb{R}$ is an integral constant.

Now, we discuss the bifurcations of the phase portraits of (6) in space (parameter β, γ). Clearly, there are three equilibrium points $O(0,0)$, $\phi_1(\phi_1,0)$, and $\phi_2(\phi_2,0)$ for (6) on the ϕ-axis, where $\phi_1=\sqrt{-\frac{\beta}{\gamma}}$, $\phi_2=-\sqrt{-\frac{\beta}{\gamma}}$, and $\beta\gamma<0$ (We consider only this case. Otherwise, the system has one equilibrium point $O(0,0)$ which is a trivial case.) By qualitative analysis [31], we have the following results:

Case 1. If $\beta>0$, $\gamma<0$, then the equilibrium point $O(0,0)$ is a center point, while the equilibrium points $P_-(-\sqrt{-\frac{\beta}{\gamma}},0)$ and $P_+(\sqrt{-\frac{\beta}{\gamma}},0)$ are saddle points of (6).

Case 2. If $\beta<0$, $\gamma>0$, then the equilibrium point $O(0,0)$ is a saddle point, while the equilibrium points $P_-(-\sqrt{-\frac{\beta}{\gamma}},0)$ and $P_+(\sqrt{-\frac{\beta}{\gamma}},0)$ are center points of (6).

According to the above results, we obtain the phase portraits of (6) (see Figs. 1 and 2). Now, we consider **Case 1** ($\beta>0$, $\gamma<0$).

Case i. Corresponding to $H(\phi,y)=-\frac{\beta^2}{4\gamma}$, we have two heteroclinic orbits of (6) connecting the equilibrium points P_- and P_+. The Hamiltonian function (7) can be written as

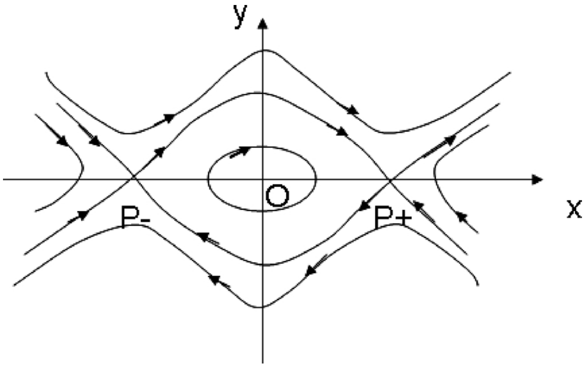

Fig. 1. Phase portrait of (6) with $\beta>0$, $\gamma<0$.

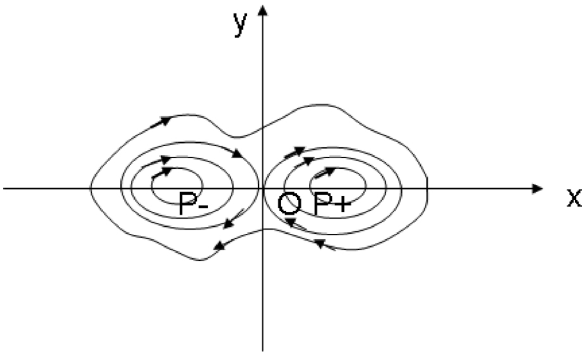

Fig. 2. Phase portrait of (6) with $\beta<0$, $\gamma>0$.

$$\begin{aligned}y^2&=-\frac{\alpha^2}{2\gamma}-\beta\phi^2-\frac{1}{2}\gamma\phi^4\\ &=-\frac{1}{2}(\phi_1-\phi)^2(\phi-\phi_2)^2.\end{aligned} \tag{8}$$

By using (8) and the first equation of (6), we obtain the following two parametric representations:

$$\phi(\xi)=\pm\sqrt{-\frac{\beta}{\gamma}}\tanh\left(\sqrt{\frac{\beta}{2}}\xi\right). \tag{9}$$

Hence, we obtain the kink and the anti-kink wave solution of (1) as

$$\begin{aligned}u(x,t)=\pm\sqrt{-\frac{\beta}{\gamma}}\tanh\left(\sqrt{\frac{\beta}{2}}k(x-ct)\right)\\ \cdot\exp(\mathrm{i}(Kx-\Omega t)).\end{aligned} \tag{10}$$

Setting $\beta=2$, $\gamma=-2$, $k=1, c=1$, then (10) reduces to $|u|=|\tanh(x-t)|$ (see Fig. 3). Here, $|u|$ is the norm of u.

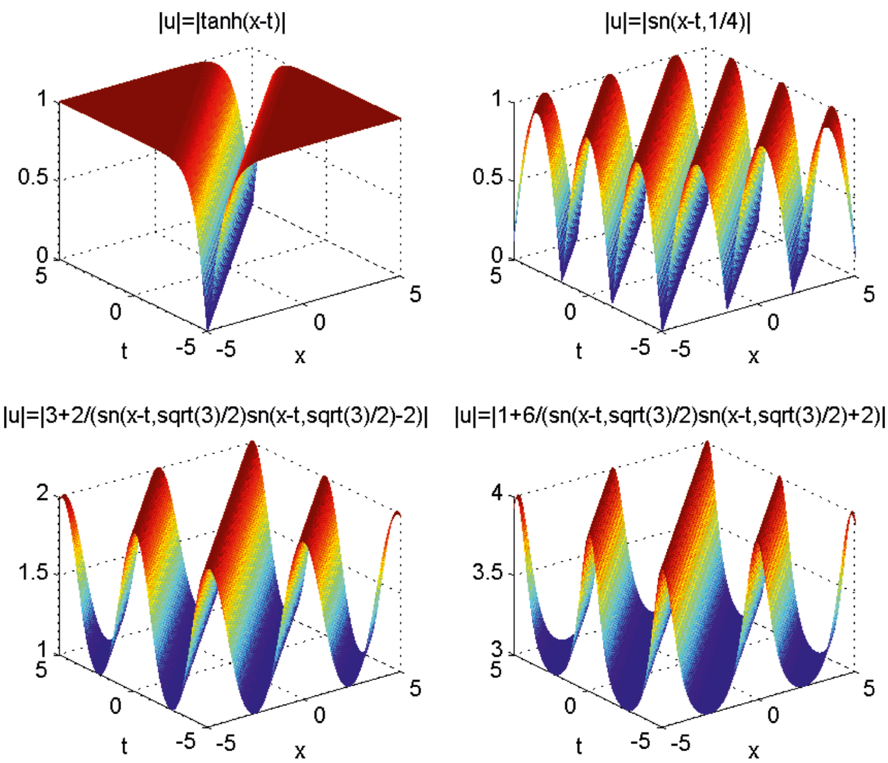

Fig. 3 (colour online). Phase portraits of (10), (13), (18), and (19), respectively.

Case ii. Corresponding to $H(\phi,y)=h$, $h\in(0,-\frac{\alpha^2}{4\gamma})$, we have a family of periodic orbits of (6) enclosing the equilibrium point $O(0,0)$, for which the function (7) can be written as

$$y^2=2h-\beta\phi^2-\frac{1}{2}\gamma\phi^4=-\frac{1}{2}(a^2-\phi^2)(b^2-\phi^2), \quad (11)$$

where $a^2=-\frac{1}{\gamma}(\beta+\sqrt{\beta^2+4h\gamma})$, $b^2=-\frac{1}{\gamma}(\beta-\sqrt{\beta^2+4h\gamma})$.

By using (11) and the first equation of (6), we obtain the following parametric representation of the family of periodic orbits:

$$\phi(\xi)=b\mathrm{sn}(\omega_1\xi,k_1), \quad (12)$$

where $\omega_1=a\sqrt{-\frac{\gamma}{2}}$, $k_1^2=\frac{b^2}{a^2}<1$. It follows that

$$u(x,t)=b\mathrm{sn}(\omega_1 k(x-ct),k_1)\exp(\mathrm{i}(Kx-\Omega t)). \quad (13)$$

This give rise to a family of periodic wave solutions of (1).

Setting $a=2$, $b=1$, $\gamma=-1$, $\omega_1=1$, $k_1=\frac{1}{4}$, $k=1$, $c=1$, then (13) reduces to $|u|=|\mathrm{sn}(x-t,\frac{1}{4})|$ (see Fig. 3).

Now, we consider **Case 2** ($\beta<0$, $\gamma>0$).

Case iii. Corresponding to $H(\phi,y)=h$, $h\in(h_2,h_1)$, where $h_1=H(\phi_1,0)$, $h_2=H(\phi_2,0)$, we have two families of periodic orbits of (6), for which the Hamiltonian function (7) can be written as

$$\begin{aligned}y^2&=2h-\beta\phi^2-\frac{1}{2}\gamma\phi^4\\&=\frac{1}{2}\gamma(r_4-\phi)(\phi-r_1)(\phi-r_2)(\phi-r_3),\end{aligned} \quad (14)$$

$$\begin{aligned}y^2&=2h-\beta\phi^2-\frac{1}{2}\gamma\phi^4\\&=\frac{1}{2}\gamma(r_4-\phi)(r_3-\phi)(r_2-\phi)(\phi-r_1),\end{aligned} \quad (15)$$

where r_i $(i=1,\ldots,4)$ can be obtained by solving the following algebraic equation with respect to

$$\phi: H(\phi,0)=h,\ \ h\in(h_2,h_1).$$

Note that for concrete parameters, we can get the values r_i by solving the algebraic equation $H(\phi,0)=h$, that is $\gamma r^4+2\beta r^2-4h=0$. By using the first equation of (6) and (14), (15), we obtain the following two parametric representations:

$$\phi(\xi) = r_3 + \frac{(r_3-r_2)(r_3-r_1)}{(r_2-r_1)\mathrm{sn}^2(\omega_2\xi,k_2)-(r_3-r_1)}, \quad (16)$$

$$\phi(\xi) = r_1 + \frac{(r_4-r_1)(r_3-r_1)}{(r_4-r_3)\mathrm{sn}^2(\omega_2\xi,k_2)+(r_3-r_1)}, \quad (17)$$

where $\omega_2 = \sqrt{\frac{(r_4-r_2)(r_3-r_1)\gamma}{8}}$, $k_2 = \sqrt{\frac{(r_4-r_3)(r_2-r_1)}{(r_4-r_2)(r_3-r_1)}} < 1$. Hence, there exist the following periodic travelling solutions of (1):

$$u(x,t) = r_3 + \frac{(r_3-r_2)(r_3-r_1)}{(r_2-r_1)\mathrm{sn}^2(\omega_2 k(x-ct),k_2)-(r_3-r_1)} \cdot \exp(\mathrm{i}(Kx-\Omega t)), \quad (18)$$

$$u(x,t) = r_1 + \frac{(r_4-r_1)(r_3-r_1)}{(r_4-r_3)\mathrm{sn}^2(\omega_2 k(x-ct),k_2)+(r_3-r_1)} \cdot \exp(\mathrm{i}(Kx-\Omega t)). \quad (19)$$

Setting $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 3$, $\gamma_4 = 4$, $\omega_2 = 1$, $k_2 = \frac{\sqrt{3}}{2}$, $k = 1$, $c = 1$, then (18) and (19) become

$$|u| = \left| 3 + \frac{2}{\mathrm{sn}^2(x-t,\frac{\sqrt{3}}{2})-2} \right|,$$

$$|u| = \left| 1 + \frac{6}{\mathrm{sn}^2(x-t,\frac{\sqrt{3}}{2})+2} \right|,$$

respectively (see Fig. 3).

Case iv. Corresponding to $H(\phi,y) = h_1$, where $h_1 = H(\phi_1,0)$, we have two homoclinic orbits of (6). The function (7) can be written as

$$\begin{aligned} y^2 =& \ 2h_1 - \beta\phi^2 - \frac{1}{2}\gamma\phi^4 \\ =& \frac{1}{2}\gamma(\phi-r_2)^2(r_3-\phi)(r_3-\phi)(\phi-r_1). \end{aligned} \quad (20)$$

By using the first equation of (6) and (20), we obtain the following two parametric representations:

$$\phi(\xi) = r_2 + \frac{2(r_3-r_2)(r_2-r_1)}{(r_3-r_1)\cos(\omega_3\xi)-(r_3-2r_2+r_1)}, \quad (21)$$

$$\phi(\xi) = r_2 + \frac{2(r_3-r_2)(r_2-r_1)}{(r_1-r_3)\cosh(\omega_3\xi,)-(r_3-2r_2+r_1)}, \quad (22)$$

where $\omega_3 = \sqrt{\frac{(r_2-r_1)(r_3-r_2)\gamma}{2}}$. Therefore, we obtain two solitary wave solutions of (1) of peak and valley type, respectively, as follows:

$$u(x,t) = r_2 + \frac{2(r_3-r_2)(r_2-r_1)}{(r_3-r_1)\cos(\omega_3 k(x-ct))-(r_3-2r_2+r_1)} \cdot \exp(\mathrm{i}(Kx-\Omega t)), \quad (23)$$

$$u(x,t) = r_2 + \frac{2(r_3-r_2)(r_2-r_1)}{(r_1-r_3)\cos(\omega_3 k(x-ct))-(r_3-2r_2+r_1)} \cdot \exp(\mathrm{i}(Kx-\Omega t)). \quad (24)$$

Setting $\gamma = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 3$, $\gamma_4 = 4$, $\omega_3 = 1$, $k = 1$, $c = 1$, then (23) and (24) become

$$|u| = \left| 2 + \frac{1}{\cos(x-t)} \right|,$$

$$|u| = \left| 2 - \frac{1}{\cos(x-t)} \right|,$$

respectively (see Fig. 4).

Case v. Corresponding to $H(\phi,y) = h$, $h \in (h_1,\infty)$, we have a family of periodic orbits of (6) enclosing three equilibrium points O, A_1, and A_2, for which the function (7) can be written as

$$\begin{aligned} y^2 &= 2h - \beta\phi^2 - \frac{1}{2}\gamma\phi^4 \\ &= \frac{1}{2}\gamma(r_2-\phi)(\phi-r_1)\left[(\phi-g_1)^2+g_2^2\right], \end{aligned} \quad (25)$$

where g_1 and g_2 are real constants. By using (25) and the first equation of (6), we obtain the following parametric representations:

$$\phi(\xi) = \frac{(r_1F-r_2G)(r_2-r_1)\mathrm{cn}(\omega_4\xi,k_3)+r_1F+r_2G}{(F-G)\mathrm{cn}(\omega_4\xi,k_3)+F+G}, \quad (26)$$

where

$$\omega_4 = \sqrt{\frac{FG\gamma}{2}}, \quad k_3^2 = \frac{(r_2-r_1)^2-(F-G)^2}{4FG},$$
$$F^2 = (r_2-g_1)^2+g_2^2, \quad G^2 = (r_1-g_1)^2+g_2^2.$$

Thus, we have the following periodic travelling wave solutions of (1):

$$u(x,t) = \frac{(r_1F-r_2G)(r_2-r_1)\mathrm{cn}(\omega_4 k(x-ct),k_3)+r_1F+r_2G}{(F-G)\mathrm{cn}(\omega_4 k(x-ct),k_3)+F+G} \cdot \exp(\mathrm{i}(Kx-\Omega t)). \quad (27)$$

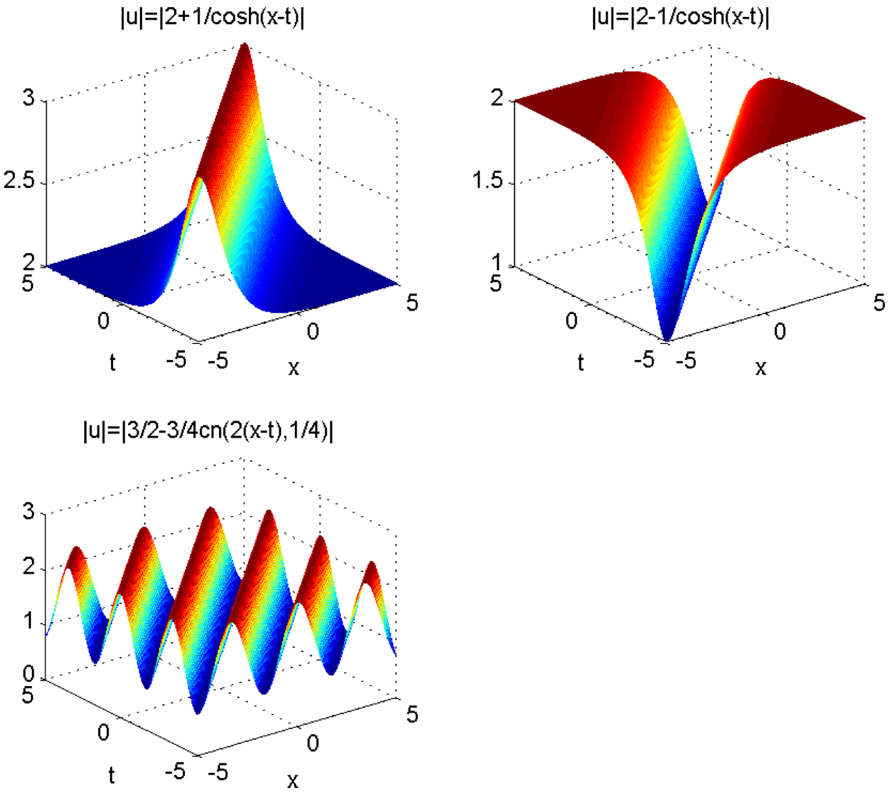

Fig. 4 (colour online). Phase portraits of (23), (24), and (27), respectively.

Setting $\gamma = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$, $g_1 = 0$, $g_2 = \sqrt{3}$, $\omega_4 = 2$, $F = G = 2$, $k_3 - \frac{1}{4}$, $k = 1, c = 1$, then (23) and (27) become

$$|u| = \left| \frac{3}{2} - \frac{3\mathrm{cn}(2(x-t), \frac{1}{4})}{4} \right|$$

(see Fig. 4).

3. Summary

In this article, in order to find the travelling wave solutions of nonlinear partial differential equations (NPDEs), we introduce the wave variables $u(x,t) = u(\xi)$ and $\xi = k(x - ct)$, where k and c are constants. So, we obtain the following ordinary differential equation (ODE): $A\phi''(\xi) + B\phi(\xi) + C\phi^3(\xi) = 0$. Then we establish the travelling wave solutions of NPDEs by using the theory of bifurcations of dynamic systems. Under the given parametric conditions, all possible representations of explicit exact solitary wave solutions and periodic wave solutions are obtained. Finally, it is worth while to mention that the method can also be applied to solve many other NPDEs in mathematical physics which will be investigated in another work.

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